Efficiency Assessment of Numerical Fractional Models in R

FMATH Group

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Introduction

- Anomalous, power-law based phenomena are ubiquitous
  - Anomalous biomechanics, Figs. 1 & 2
  - Eddies in turbulent flows, Fig. 3
  - Anomalous diffusion in transient media

Figure 1. Viscoelastic behavior of human tissue.

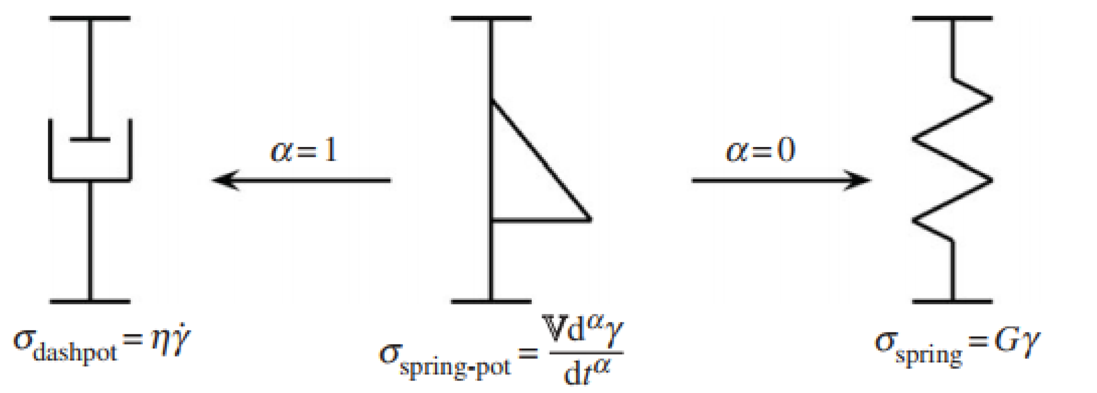

Figure 2. Physical interpretation of fractional derivatives as interpolation operators in anomalous materials [1].

Figure 3. Non-Brownian particle motion due to particle trapping in turbulent eddies.

- Fractional derivatives capture the power-law history of anomalous phenomena, shown in Eqn. 1

$$ {}^{RL}_{0}\mathcal{D}^{\nu}_{t} u(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t (t-s)^{-\nu} u(s)\, ds \quad (1)$$

Kernel captures power law

Integration captures history

- Low CPU time and high accuracy are desired
- Benchmark criterion required to quantify computational efficacy
- Thus, we develop an assessment framework to evaluate numerical schemes
- R platform works well with large sets of data

Methods

- Three methods were benchmarked with Eqn. 2, a standard FODE
  - Petrov-Galerkin Spectral Method (PGSM) [2]
  - Finite Difference Method (FDM) [2]
  - Finite Element Method (FEM) [3]

$$ {}^{RL}_{0}\mathcal{D}^{\nu}_{t} u(t) = f(t), u(0) = 0 \quad (2)$$

- Tested with method of fabricated solution for CPU time and error,

$$ \hat{E}_{L_2} = \frac{||u^{ex}(t) - u^{app}(t)||}{||u^{ex}(t)||}$$

- Schemes are applied to anomalous Maxwell material stress-strain behavior via Eqn. 3 [1]

$$ \mathbb{V}\, {}^{RL}_{0}\mathcal{D}^{\nu}_{t} \epsilon(t) = \sigma(t) + \frac{\mathbb{V}}{\mathbb{G}}\, {}^{RL}_{0}\mathcal{D}^{\nu-\mu}_{t} \sigma(t) \quad (3)$$

where $\epsilon(t)$ and $\sigma(t)$ are the strain and stress of the material

Results

- Fig. 4 shows a sample fit of the test benchmark function ${}^{RL}_{0}\mathcal{D}^{0.2}_{t} t^{5.1} = \frac{\Gamma(6.1)}{\Gamma(5.9)} t^{4.9}$ using PGSM

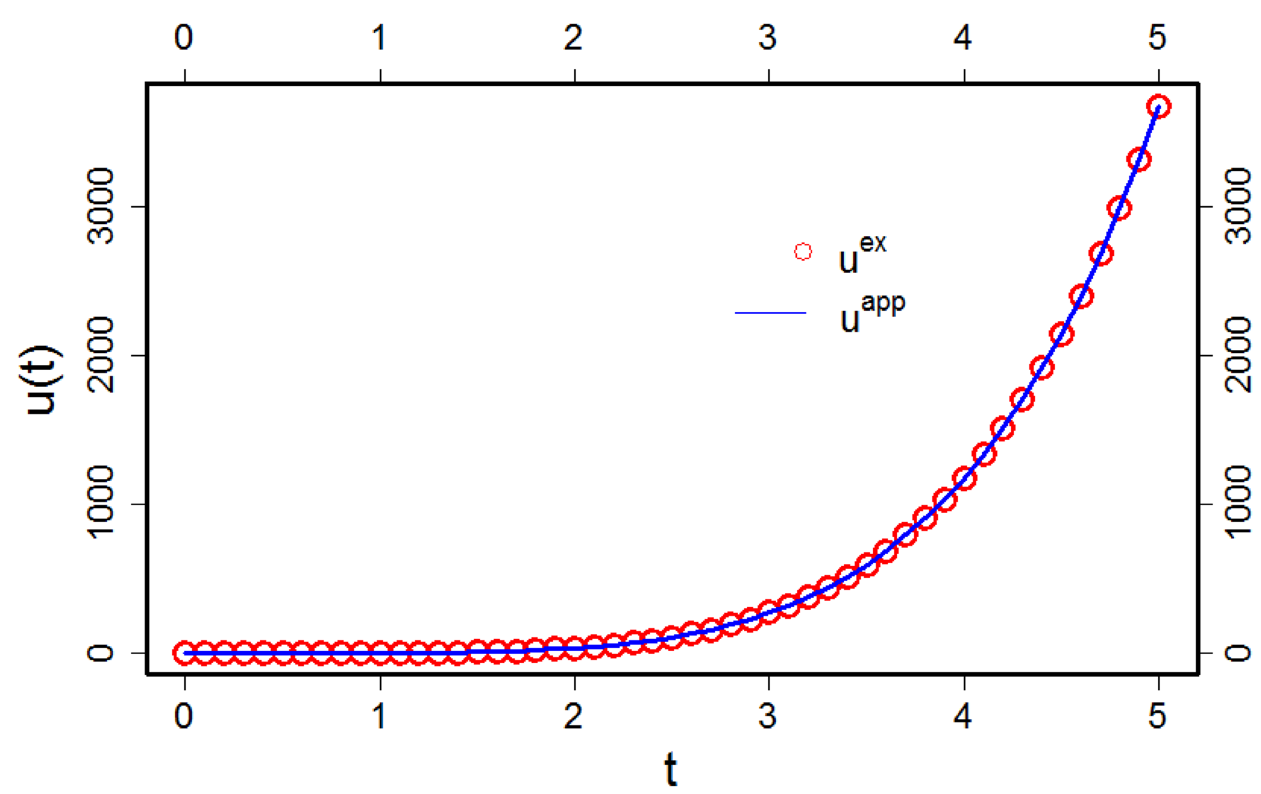

Figure 4. Simulation of benchmark equation ${}^{RL}_{0}\mathcal{D}^{0.2}_{t} t^{5.1} = \frac{\Gamma(6.1)}{\Gamma(5.9)} t^{4.9}$ using PGSM with 6 modes.

- Fig. 5 shows the convergence rates for PGSM, FDM, and FEM

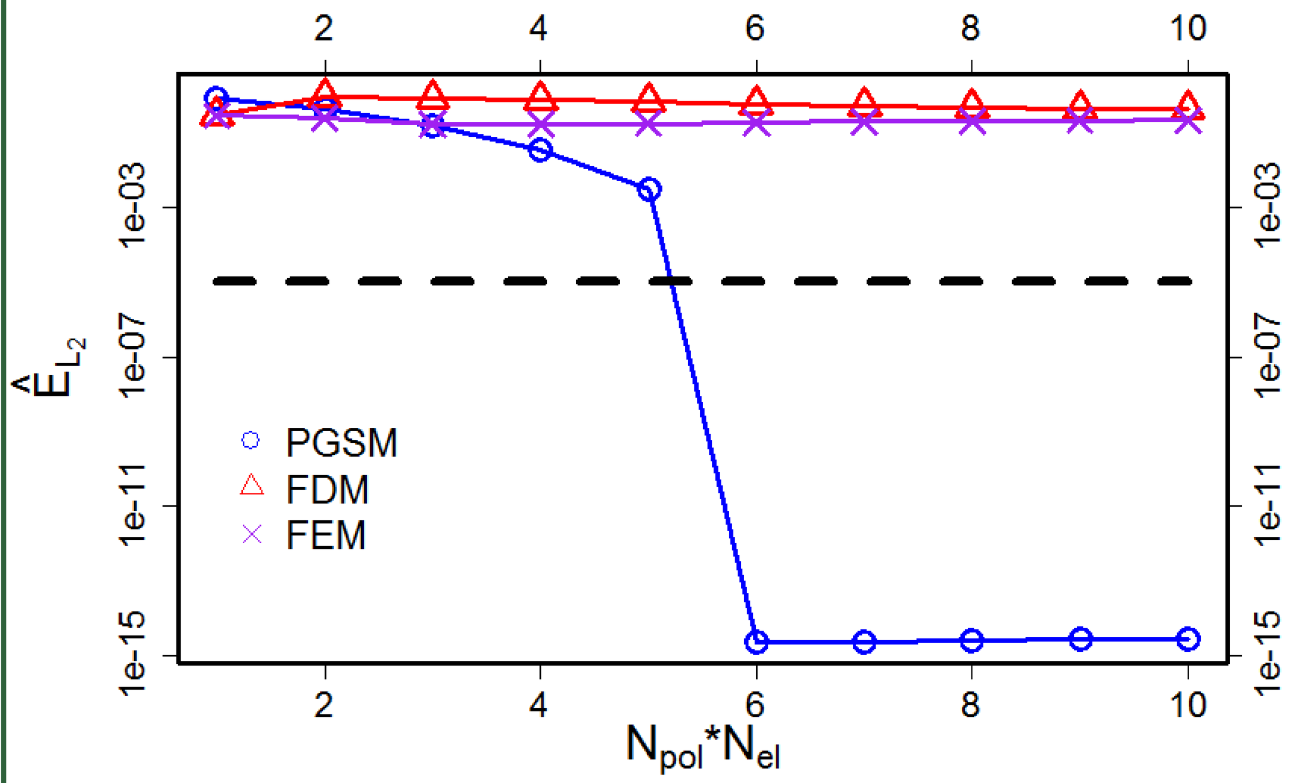

Figure 5. Rate of convergence for ${}^{RL}_{0}\mathcal{D}^{0.2}_{t} t^{5.1} = \frac{\Gamma(6.1)}{\Gamma(5.9)} t^{4.9}$. PGSM demonstrates absolute convergence at 6 terms.

- Table 1 lists CPU times required to reach $\mathcal{O}(10^{-5})$ error for each method

Table 1. Errors for various methods at $N_{pol} * N_{el} = 6$

Method	$\hat{E}_{L_2}$ at $N_{pol} * N_{el} = 6$
PGSM	2e-15
FDM	6e-1
FEM	2e-1

- Fig. 6. shows the simulations of fractional Maxwell material, Eqn. 2, using PGSM

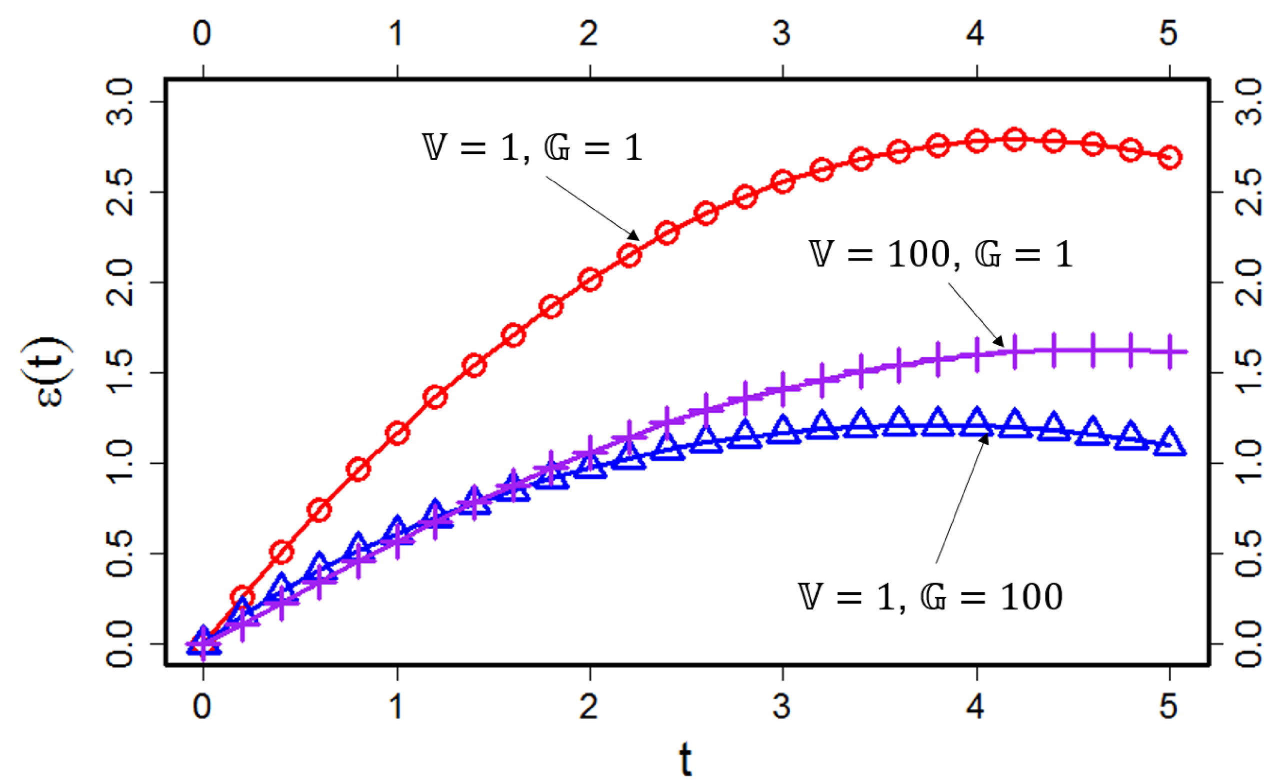

Figure 6. Simulations of strain, $\epsilon(t)$, versus test stress function, $\sigma(t) = \sigma_0 \left[\frac{t^{\nu-\mu}}{T} - \frac{t^{1+\nu-\mu}}{T}\right]$, with $\nu = 0.9$, $\mu = 0.7$, and $\sigma_0 = 1$ and various $\mathbb{V}$ and $\mathbb{G}$

Discussion

- Framework records evaluation criterion for schemes, listed in Fig. 5 and Table 1
- Table 2 lists insight into beneficial algorithm mechanics

Table 2. Key features for various methods

Method	Key Feature
PGSM	Diagonal linear system
FDM	Generalizable form
FEM	Adaptive to irregularities

- Table 3 lists scheme pros and cons
- Data from assessment framework supports intuitive pros and cons of algorithms

Table 3. Pros and cons for various methods

Pros	Cons
PGSM	
• Captures singular & smooth behavior • High, fast accuracy	• Hard to implement • Limited to smooth functions
FDM	
• Generalizable • Easy to Implement	• Large history iteration • High accuracy unfeasible
FEM	
• Adaptive to irregular domains or singularities	• Large history matrix

Future Work

- Extension to fractional diffusion equation numerical schemes
- Implement data fitting performance metrics for large data sets

References

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