Efficiency Assessment of Fractional Models in R

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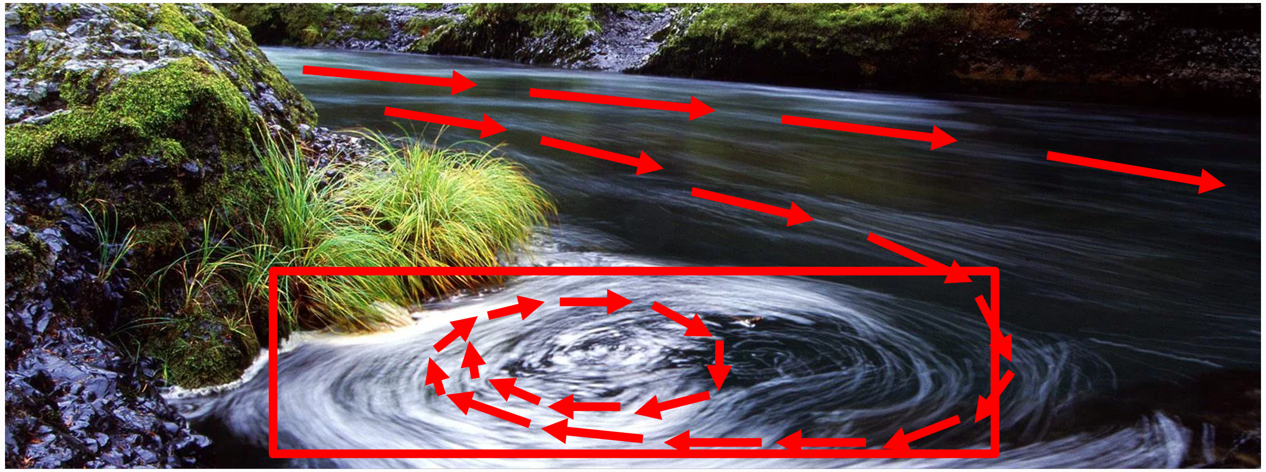
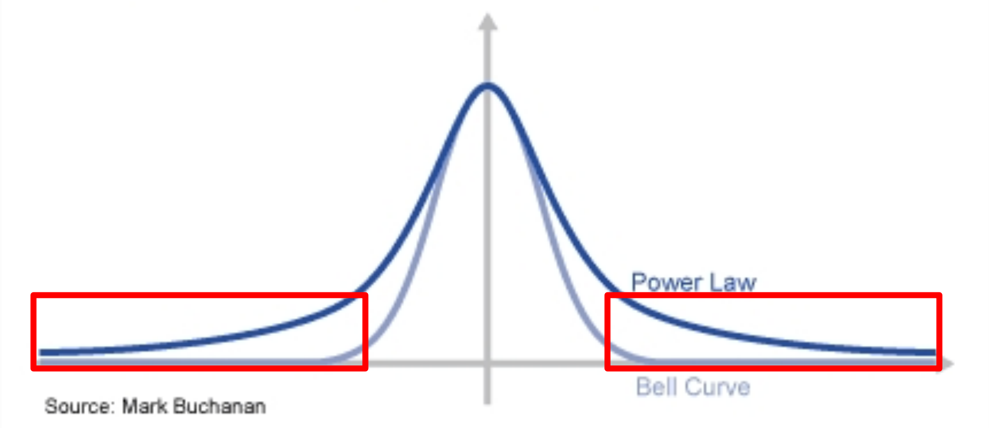
Fractional Models in Nature





Non-standard CDF with power-law tails

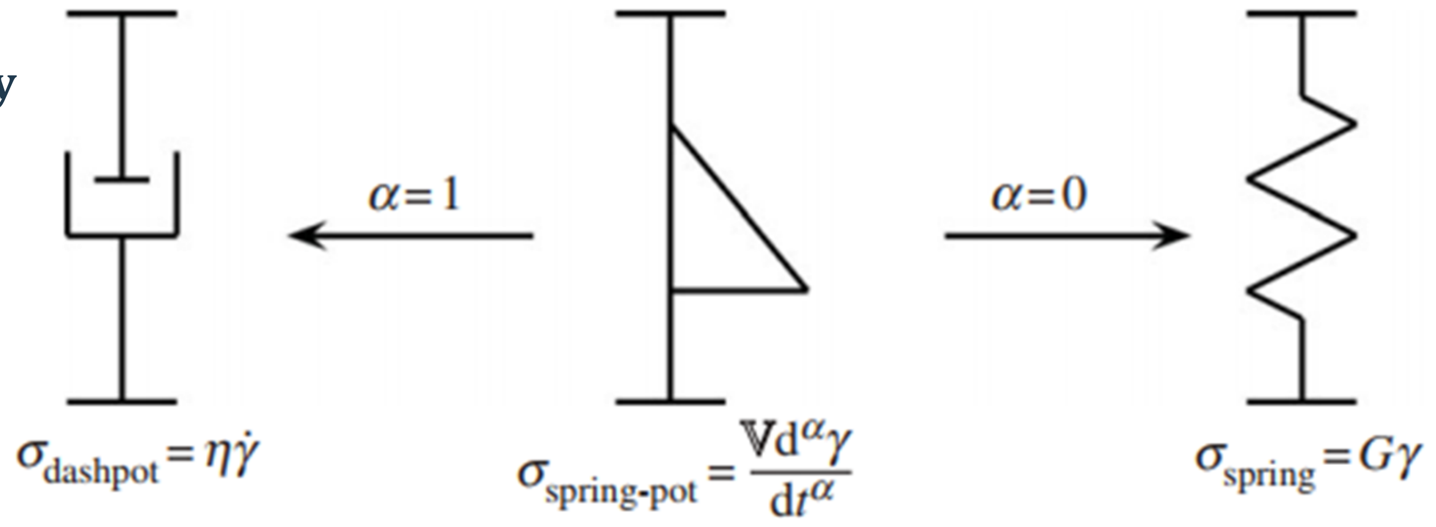
Fractional Derivative

$$ {}^{RL}_{0}\mathcal{D}_t^{\nu} u(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t (t-s)^{-\nu} u(s)\, ds \quad (1) $$

Kernel captures power law

Integration captures history/non-locality

Interpolation operator



Methods in R

Projection (p-refinement)

$$ {}_{0}^{RL}\mathcal{D}_{t}^{\nu} u(t) = f(t), u(0) = 0 $$

1. Expansion into P solution functions

$$ u(t) = \sum_{p=1}^{P} \hat{u}_p \phi_p(t) = \sum_{p=1}^{P} u_p(t) $$

2. Project problem onto test functions

$$ \int_0^T {}_{0}^{RL}\mathcal{D}_{t}^{\nu}[u_p(t)] v_j(t)\, dt = \int_0^T f(t) v_j(t)\, dt $$

3. Solve for coefficients

$$ \mathbf{S}\vec{u} = \mathbf{F} $$

Discretization (h-refinement)

$$ {}_{0}^{RL}\mathcal{D}_{t}^{\nu} u(t) = f(t), u(0) = 0 $$

1. Define discretization scheme

$$ {}_{0}^{RL}\mathcal{D}_{t}^{\nu} u(t) \cong \frac{\sum_{n=0}^{T/h} (-1)^n \binom{\alpha}{n} u(t - nh)}{h^{\nu}} $$

2. Define initial condition

$$ u(0) = 0 $$

3. Iterate forward

Petrov-Galerkin Spectral Method

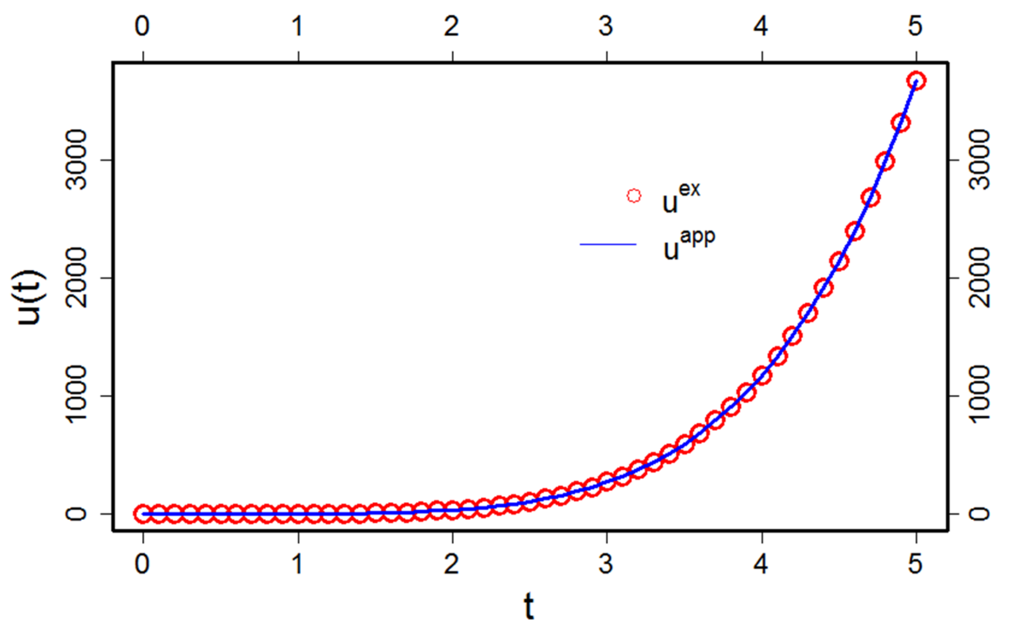
p-refinement method

$$u(t) = \sum_{p=1}^{P} \hat{u}_p \big(1+\xi(t)\big)^{\frac{\nu}{2}} P_{p-1}^{-\frac{\nu}{2},\frac{\nu}{2}}\big(\xi(t)\big)$$

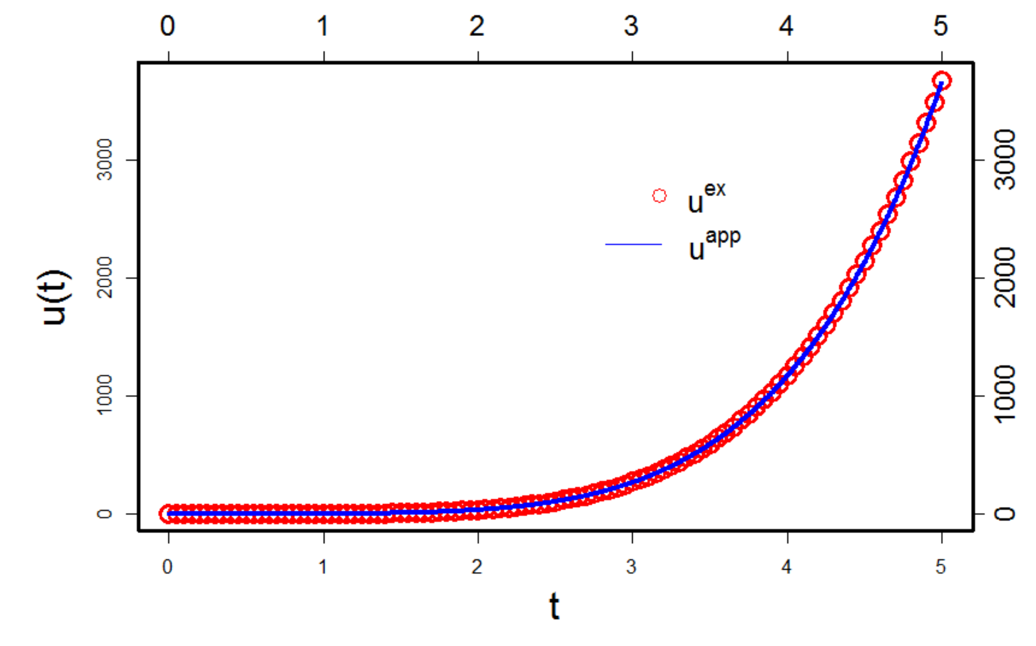
$$v_j(t) = \big(1-\xi(t)\big)^{\frac{\nu}{2}} P_{j-1}^{\frac{\nu}{2},-\frac{\nu}{2}}\big(\xi(t)\big)$$

$$\mathbf{S}\vec{u} = \mathbf{F}$$

S is diagonal



Finite Difference Method

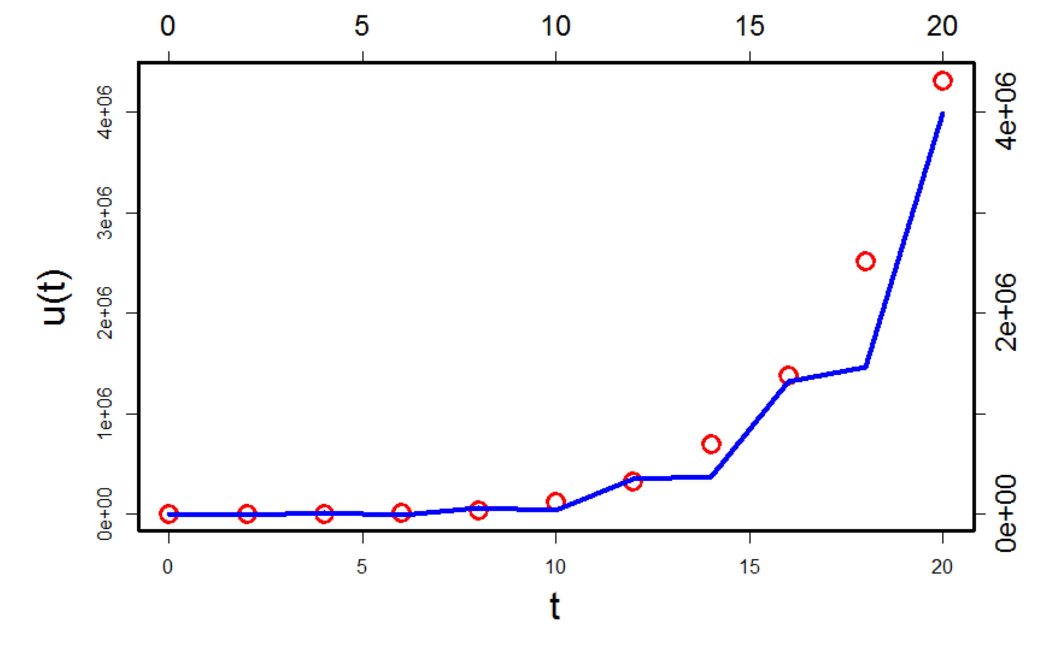


h-refinement method

$$u(t_{k+1}) = (1-b_1)u(t_k) + \sum_{j=1}^{k-1}(b_j - b_{j+1})u(t_{k-j}) + b_k u(t_0) + \Delta t^{\nu}\Gamma(2-\nu)f(t_{k+1})$$

Simple iteration

Finite Element Method



hp-refinement method

$$\sum_{e=1}^{\epsilon-1}\sum_{p=1}^{P}\vec{u}_p^e H_{\epsilon,e,p} + \sum_{p=1}^{P}\vec{u}_p^\epsilon M_{\epsilon,p} = F_\epsilon$$

$$\mathbb{M}_G\vec{u}_G = \mathbb{F}_G$$

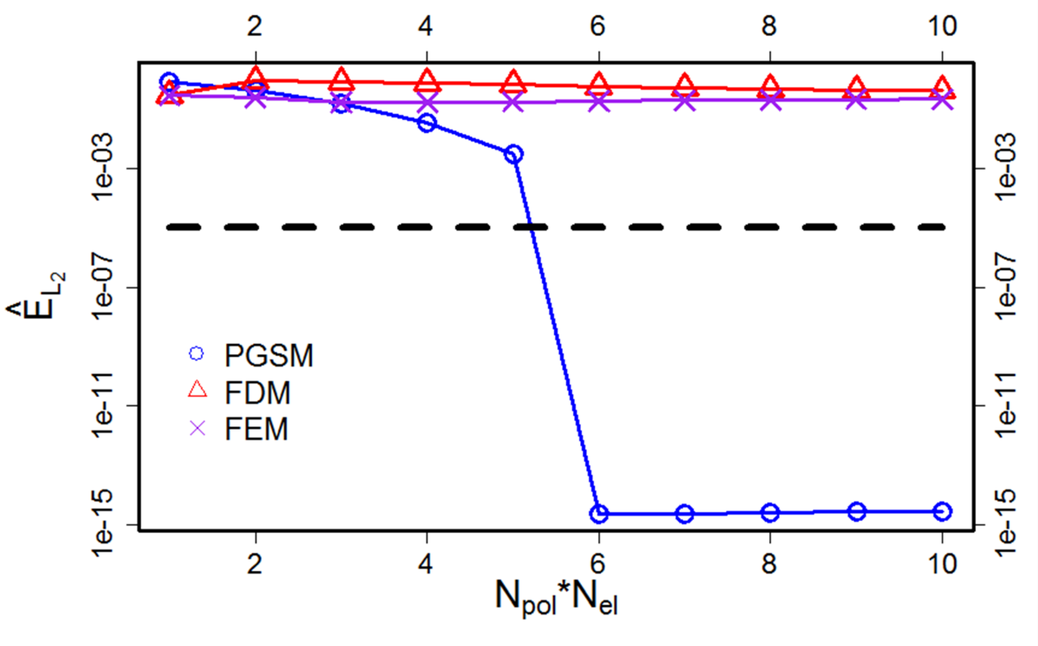
Originally for elliptical problems

Convergence issues

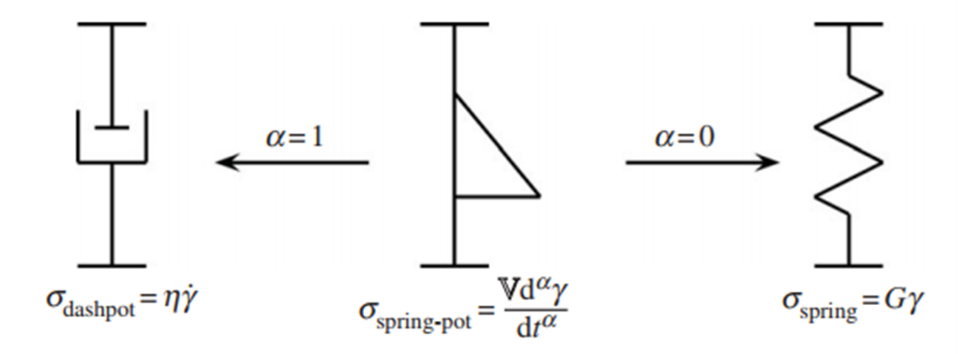
Comparison

Pros and cons for various methods

Pros	Cons
PGSM	
• Captures singular & smooth behavior • High, fast accuracy	• Hard to implement • Limited to smooth functions
FDM	
• Generalizable • Easy to Implement	• Large history iteration • High accuracy unfeasible
FEM	
• Adaptive to irregular domains or singularities	• Large history matrix

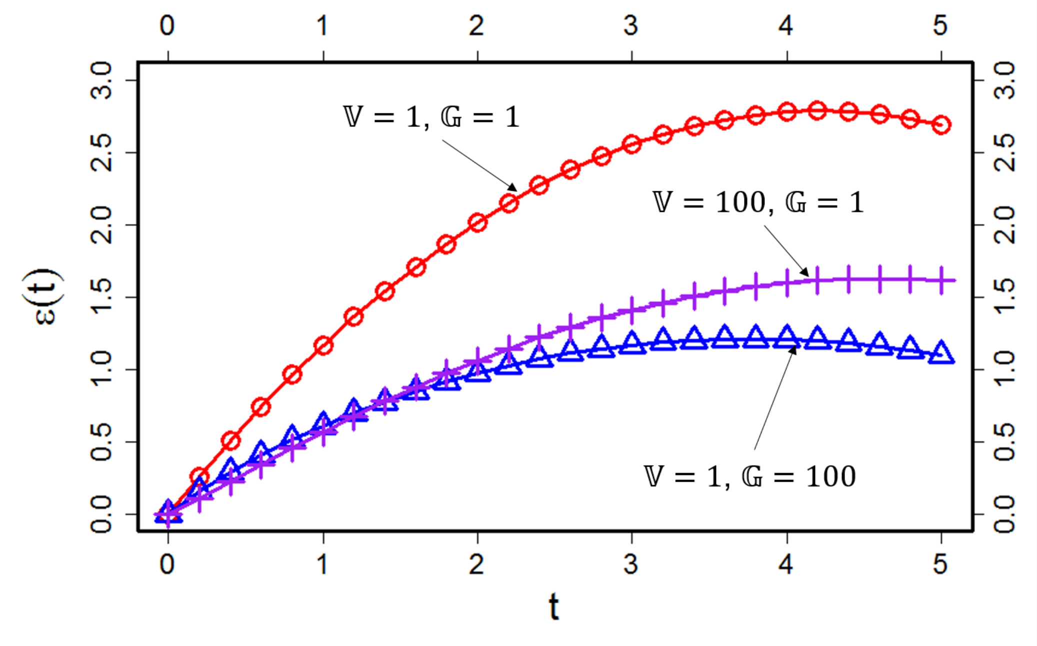


Fractional Maxwell Application



$$ {}^{RL}_{0}\mathcal{D}_t^{\nu}\epsilon(t) = \frac{1}{\mathbb{V}}\sigma(t) + \frac{1}{\mathbb{G}}\,{}^{RL}_{0}\mathcal{D}_t^{\nu-\mu}\sigma(t) $$

Resembles ${}^{RL}_{0}\mathcal{D}_t^{\nu}u(t) = f(t)$



Discussion

Selection of method depends on application

- Data with power-law singularity/models → PGSM
- Non-intuitive data/models → FDM
- Fragmented data/domains → FEM

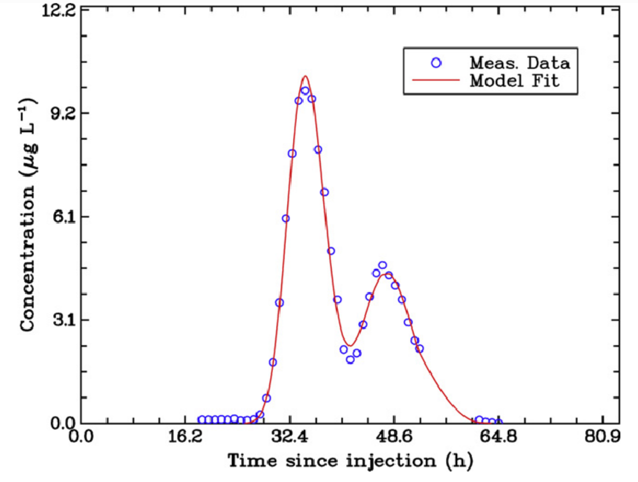
Key features for various methods

Method	Key Feature
PGSM	Diagonal linear system
FDM	Generalizable form
FEM	Adaptive to irregularities

Future Work

- Extension to fractional advection-dispersion equation

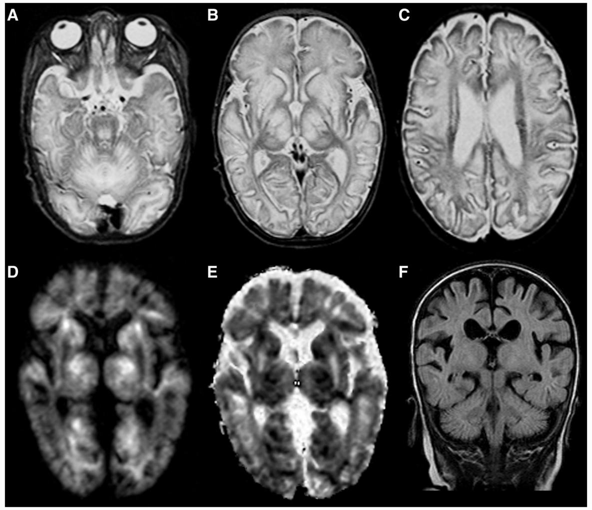
$$ {}_0\mathcal{D}_t^{2\tau}u + \sum_{i=1}^{d}\left[(1-\theta_i)\,{}_{a_i}\mathcal{D}_{x_i}^{2\mu_i} + (\theta_i)\,{}_{x_i}\mathcal{D}_{b_i}^{2\mu_i}\right]u = \sum_{i=1}^{d}\left[(1-\phi_i)\,{}_{a_i}\mathcal{D}_{x_i}^{2\nu_i} + (\phi)\,{}_{x_i}\mathcal{D}_{b_i}^{2\nu_i}\right]u - \gamma u + f $$



- Inclusion of error estimates against observable data

- Take advantage of R platform for observable data comparisons

- Fine tune CPU time analysis



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