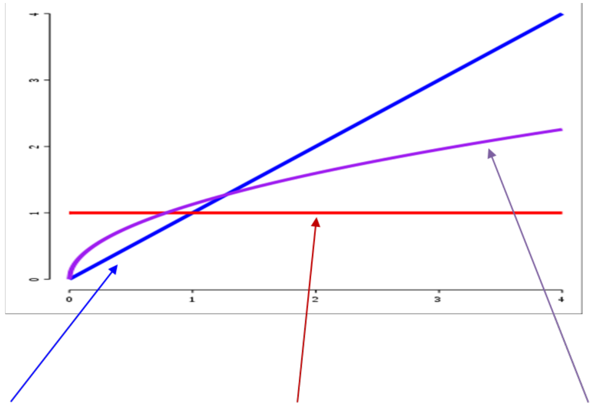
A Computational Model for Anomalous Diffusion in Bio-Tissues

STUDENT: SEAN CONNELLY

ADVISOR: DR. MOHSEN ZAYERNOURI

The Fractional Derivative

- Foundation of fractional calculus.
- Interpolates between integer orders of derivatives.
- Allows for a wider range of flux values; very accurate for anomalous diffusion in nature[3].



Caputo fractional derivative

$$({}_0^C\mathcal{D}_t^{\alpha} T)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{T'(x,s)}{(t-s)^{\alpha}}\, ds, \quad 0 < \alpha < 1$$

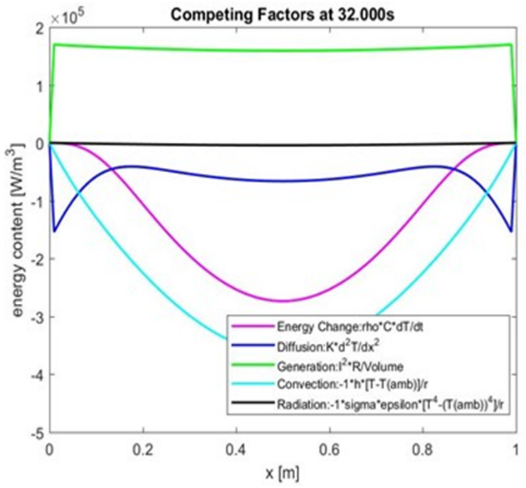
- Uses a power law kernel.
- The integral accounts for the history up to the beginning of the process.

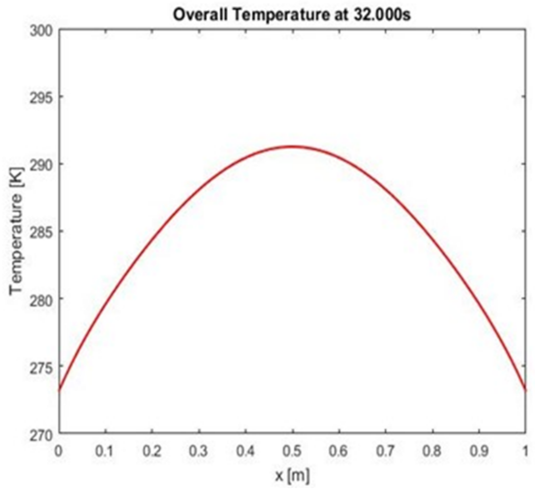
The One-Dimensional Heat Equation

- Basis for modeling diffusion within the brain.
- Initially an integer-order equation.

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}K + \frac{I^2 R}{A_C \Delta x} - \frac{2h}{r}[T - T_{amb}] - \frac{2\sigma\epsilon}{r}[T^4 - T_{amb}^4]$$

Transient Term | Diffusion Term | Generation Term | Convection Term | Radiation Term





Forward-Euler finite difference in time:

$$\frac{\partial T}{\partial t} = \frac{T_{x_j}^{t_{n+1}} - T_{x_j}^{t_n}}{\Delta t}$$

Central finite-difference in space:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\left[T_{x_j-1}^{t_n} - 2T_{x_j}^{t_n} + T_{x_j+1}^{t_n}\right]}{\Delta x^2}$$

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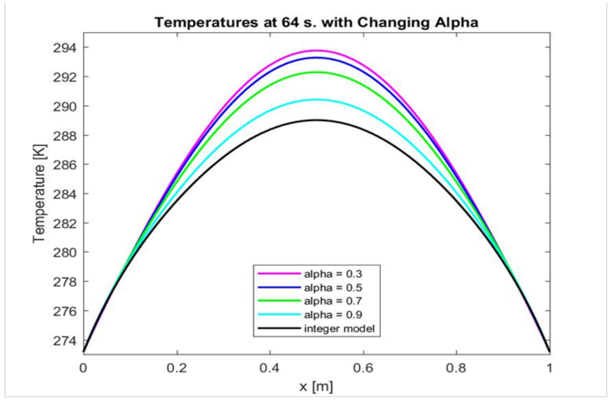
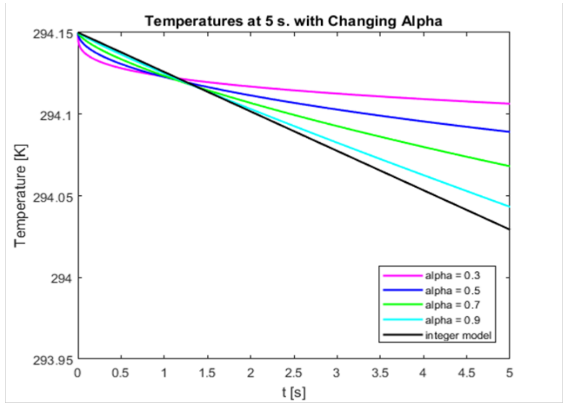
The Time-Fractional Heat Equation

- Incorporates the fractional-order derivative into the 1D heat equation.

$$\rho C \, {}_0^C\mathcal{D}_t^\alpha (T_{x_j}^{t_n}) = K \frac{\partial^2 T}{\partial x^2} + f(x,t)$$



Observe that lower values of α result in a slower relaxation rate, and thus a higher overall temperature at a given time step.



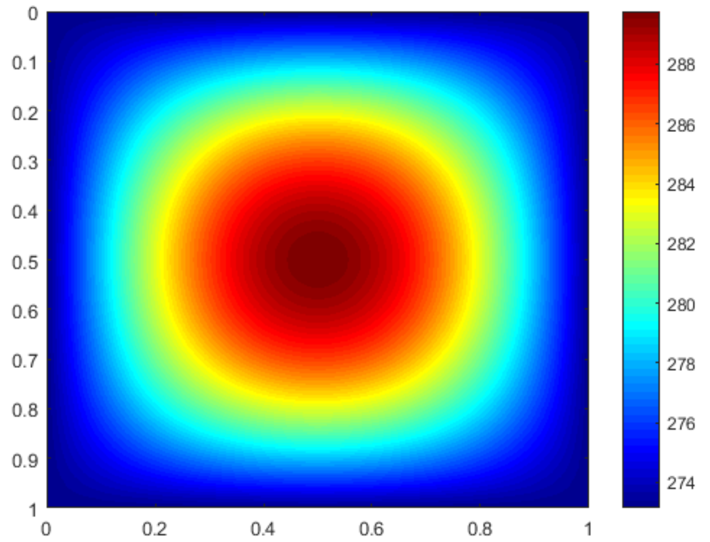
We initially observe a period of “super-diffusivity” until around 1.3s, in which the lower values of α result in a higher rate of diffusion.

The Two-Dimensional Heat Equation

- Incorporates diffusion in the y-direction.
- First stage in obtaining an accurate cross-sectional model.

$$T(x,y,0) = 273.15 + 21\left(\sin\left(\frac{\pi x}{L_x}\right)\sin\left(\frac{\pi y}{L_y}\right)\right)$$

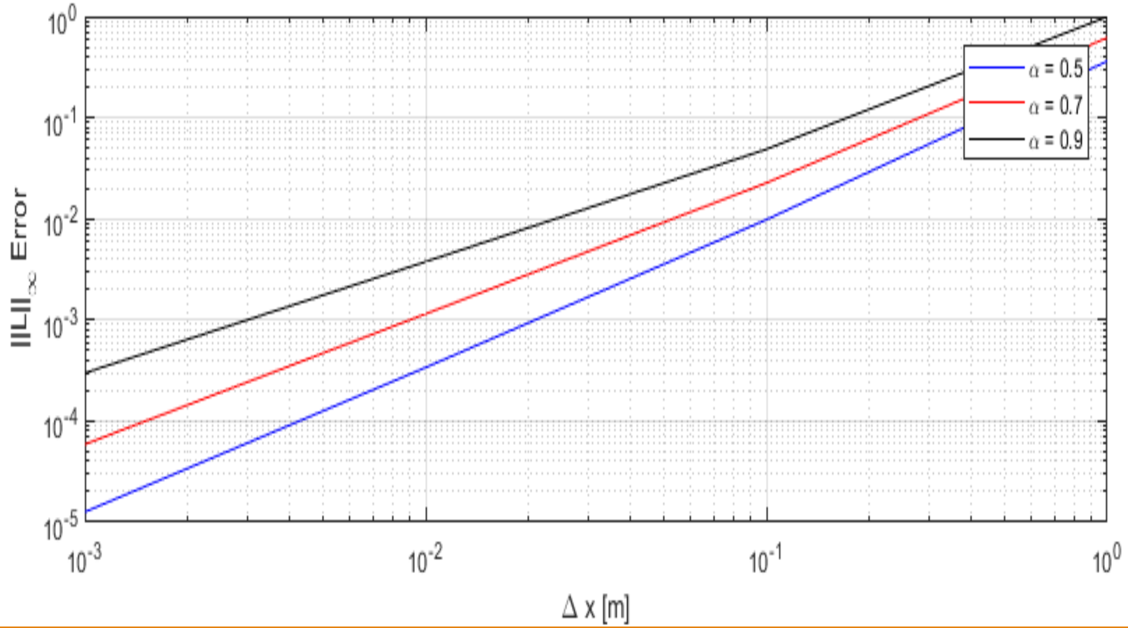
$$^C_0\mathcal{D}^\alpha_t(T^n_{i,j}) = D\frac{\partial^2 T}{\partial x^2} + D\frac{\partial^2 T}{\partial y^2} + f(x,y,t)$$



Convergence Analysis

- Uses a fabricated solution to test the accuracy of our discretization scheme.
- Converges with on a log/log graph with a slope of (2-α).

$$T^{\delta}(x,y,t) = T_0 \sin\left(\frac{\pi x}{L_x}\right)\sin\left(\frac{\pi y}{L_y}\right)\left(\frac{t}{t_F}\right)^{q}$$

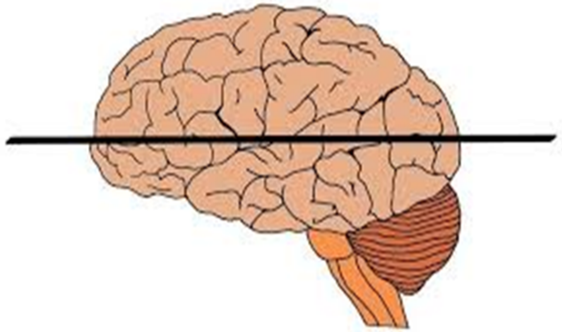


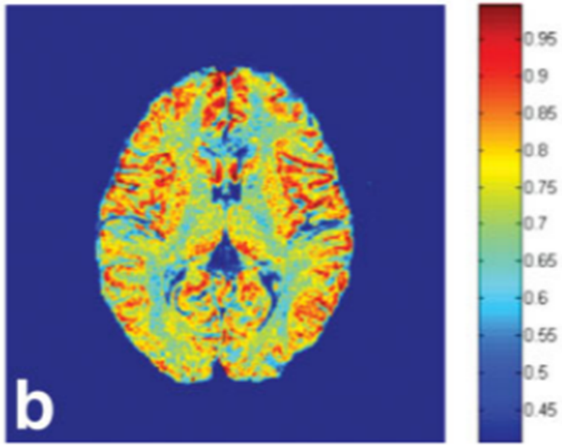
Future Work

- Discretize a time- *and* space-fractional, 2D model.

- Simulate diffusion within a cross-section of the brain, using “hot spots” of activity.

- Generalize the 2D model into a complete, 3D model.





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Works Cited

[1] Magin, Richard L., et al. "Anomalous diffusion expressed through fractional order differential operators in the Bloch–Torrey equation." Journal of Magnetic Resonance 190.2 (2008): 255-270.

[2] Magin, Richard, Xu Feng, and Dumitru Baleanu. "Solving the fractional order Bloch equation." Concepts in magnetic resonance. Part A, Bridging education and research 34.1 (2009): 16.

[3] Zhou, Xiaohong Joe, et al. "Studies of anomalous diffusion in the human brain using fractional order calculus." Magnetic resonance in medicine 63.3 (2010): 562-569.